Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 5, pp. 764–773, 2004

INDENTATION OF A PUNCH WITH A FINE-GRAINED BASE INTO AN ELASTIC FOUNDATION

I. I. Argatov

UDC 539.3

The linear contact problem for a system of small punches located periodically on a part of the boundary of an elastic foundation is studied. An averaged contact problem is derived using the Marchenko–Khruslov averaging theory. An asymptotic formula is obtained for the translational capacity of a smooth punch with a fine-grained flat base.

Key words: *discrete contact, averaged contact pressure, fine-grained base.*

1. Formulation of the Problem. We consider a bounded domain G on the plane $\mathbb{R}^2$ with a smooth boundary. Let l be the characteristic dimension of the region, and let ω be a plane region bounded by a smooth contour inside the square $K = (-l/2, l/2) \times (-l/2, l/2)$. We set

$$\omega^{ij}(\varepsilon) = \{(x_1, x_2)\colon \varepsilon^{-2}(x_1 - i\varepsilon l, x_2 - j\varepsilon l) \in \omega\} \qquad (i, j \in \mathbb{Z}), \tag{1.1}$$

where ε is a small positive parameter. We denote by Γ_ε the set of all spots $\omega^{ij}(\varepsilon)$ located in the domain G.

In accordance with the Papkovich–Neuber representation (see, e.g., [1]), the linear contact elastic problem of a punch with a smooth flat base that occupies the set Γ_ε and is indented into an elastic half-space to unit depth reduces to the following mixed boundary-value problem of the theory of harmonic functions:

$$\Delta_x u^\varepsilon(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3_+ = \{\boldsymbol{x} = (x_1, x_2, x_3)\colon x_3 > 0\}; \tag{1.2}$$

$$u^\varepsilon(\boldsymbol{x}', 0) = 1, \qquad \boldsymbol{x}' = (x_1, x_2) \in \Gamma_\varepsilon; \tag{1.3}$$

$$\frac{\partial u^\varepsilon}{\partial x_3}(\boldsymbol{x}', 0) = 0, \qquad \boldsymbol{x}' \in \mathbb{R}^2 \setminus \overline{\Gamma}_\varepsilon; \tag{1.4}$$

$$u^\varepsilon(\boldsymbol{x}) = o(1), \qquad |\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \to \infty. \tag{1.5}$$

In this case, the displacement-vector components $\boldsymbol{U}^\varepsilon = (U_1^\varepsilon, U_2^\varepsilon, U_3^\varepsilon)$ for the points of the elastic half-space are expressed in terms of the potential u^ε by Belyaev's formulas [2]

$$U_i^\varepsilon(\boldsymbol{x}) = \alpha\Big[(\alpha^{-1} - 1) \int\limits_{x_3}^{\infty} \frac{\partial u^\varepsilon}{\partial x_3}(x_1, x_2, z)\, dz - x_3\, \frac{\partial u^\varepsilon}{\partial x_i}(\boldsymbol{x})\Big], \quad i = 1, 2; \tag{1.6}$$

$$U_3^\varepsilon(\boldsymbol{x}) = u^\varepsilon(\boldsymbol{x}) - \alpha x_3\, \frac{\partial u^\varepsilon}{\partial x_3}(\boldsymbol{x}), \qquad \alpha = [2(1 - \nu)]^{-1}. \tag{1.7}$$

Accordingly, the contact pressure on the boundary of the elastic semi-infinite body exerted by the punch Γ_ε is given by

$$p^\varepsilon(x_1, x_2) = -\frac{E}{2(1 - \nu^2)}\, \frac{\partial u^\varepsilon}{\partial x_3}(\boldsymbol{x}', 0) \qquad (\boldsymbol{x}' \in \Gamma_\varepsilon), \tag{1.8}$$

Makarov State Marine Academy, St. Petersburg 199026; argatov@home.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 5, pp. 176–186, September–October, 2004. Original article submitted December 10, 2003.

 0021-8944/04/4505-0764

where E and ν are Young's modulus and Poisson's ratio, respectively. We note that the maximum principle for harmonic functions (see, e.g., [3]) implies the inequality

$$\frac{\partial u^\varepsilon}{\partial x_3}(\boldsymbol{x}', 0) < 0 \qquad (\boldsymbol{x}' \in \Gamma_\varepsilon), \tag{1.9}$$

which ensures a positive contact pressure (1.8).

For $|\boldsymbol{x}| \to \infty$, the following asymptotic formula [which refines formula (1.5)] holds:

$$u^\varepsilon(\boldsymbol{x}) = \boldsymbol{c}^\varepsilon |\boldsymbol{x}|^{-1} + O(|\boldsymbol{x}|^{-2}).$$

Here $\boldsymbol{c}^\varepsilon = \operatorname{cap}(\boldsymbol{\Gamma}_\varepsilon)$ is the harmonic capacity of the set $\boldsymbol{\Gamma}_\varepsilon = \{\boldsymbol{x}\colon \boldsymbol{x}' \in \overline{\Gamma}_\varepsilon,\ x_3 = 0\}$ (see, e.g., [4, 5]), given by

$$\boldsymbol{c}^\varepsilon = -\frac{1}{2\pi} \iint\limits_{\Gamma_\varepsilon} \frac{\partial u^\varepsilon}{\partial x_3}(\boldsymbol{x}', 0)\, d\boldsymbol{x}' = \frac{1}{2\pi} \iiint\limits_{\mathbb{R}^3_+} |\nabla_x u^\varepsilon(\boldsymbol{x})|^2\, d\boldsymbol{x}. \tag{1.10}$$

We note that $\boldsymbol{\Gamma}_\varepsilon$ is a plane three-dimensional set and $\overline{\Gamma}_\varepsilon$ is its two-dimensional image on the plane $x_3 = 0$.

In accordance with the electrostatic analogy [1], the quantity $\boldsymbol{c}^\varepsilon$ is called the translational capacity of a punch with a smooth flat base Γ_ε [6]. For the punch Γ_ε indented to a depth δ_0, the magnitude of the contact force is given by

$$F_3 = \iint\limits_{\Gamma_\varepsilon} p^\varepsilon(x_1, x_2)\, d\boldsymbol{x}'. \tag{1.11}$$

Using formulas (1.8) and (1.10), we obtain

$$F_3 = \frac{\pi E}{1 - \nu^2}\, \boldsymbol{c}^\varepsilon \delta_0.$$

Let us study the behavior of the solution $u^\varepsilon(\boldsymbol{x})$ of problem (1.2)–(1.5) for $\varepsilon \to 0$ using the Marchenko–Khruslov theory [7, 8]. It should be noted that the approach of [7, 8] does not require periodicity of the set Γ_ε. However, the doubly periodic location of the contact spots $\omega^{ij}(\varepsilon)$ in the domain G simplifies the calculations substantially without loss of generality.

The contact problem for a system of closely spaced small punches located periodically in a bounded region on the surface of an elastic half-space was studied in [9] using the following assumptions [compare with (1.1)]:

$$\omega^{ij}(\varepsilon) = \{\boldsymbol{x}'\colon \varepsilon^{-1}(x_1 - i\varepsilon l, x_2 - j\varepsilon l) \in \omega\}, \qquad i, j \in \mathbb{Z}. \tag{1.12}$$

In other words, the diameters of the contact spots and the distance between neighboring punches were assumed in [9] to be of the same order.

In the case considered (1.1), unlike in [9], the diameters of the contact spots are assumed to be small compared to the distance between them. As in [9], we obtain the following estimate for the number of contact spots N in the set Γ_ε for $\varepsilon \to 0$:

$$N \sim |G|/(\varepsilon^2 l^2), \tag{1.13}$$

where $|G|$ is the area of the domain G.

The contact problems for a finite number of small punches were Extensively studied (see, e.g., [1, 10] and a survey [11]). The solution of the contact problem (1.2)–(1.5) can be used to solve some contact problems of tribology [12].

2. Averaged Problem. Extending the function $u^\varepsilon(\boldsymbol{x})$ to the entire space $\mathbb{R}^3$ for evenness and taking into account the homogeneous boundary condition (1.4), we obtain the relations

$$\Delta_x u^\varepsilon(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \boldsymbol{\Gamma}_\varepsilon; \tag{2.1}$$

$$u^\varepsilon(\boldsymbol{x}', 0) = 1, \qquad \boldsymbol{x} \in \boldsymbol{\Gamma}_\varepsilon. \tag{2.2}$$

In [7, 8], it was found that as $\varepsilon \to 0$, the solution $u^\varepsilon(\boldsymbol{x})$ of problem (2.1), (2.2), (1.5) converges to a function $u^0(\boldsymbol{x})$ that is the solution of the problem (see [8, Theorem 1.4]):

$$\Delta_x u^0(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \boldsymbol{G}; \tag{2.3}$$

$$u^0(\boldsymbol{x}', 0+) = u^0(\boldsymbol{x}', 0-) = u^0(\boldsymbol{x}', 0);$$

$$\frac{\partial u^0}{\partial x_3}(\boldsymbol{x}', 0+) - \frac{\partial u^0}{\partial x_3}(\boldsymbol{x}', 0-) = 4\pi c_\Gamma (u^0(\boldsymbol{x}', 0) - 1), \qquad \boldsymbol{x} \in \boldsymbol{G}; \tag{2.4}$$

$$u^0(\boldsymbol{x}) = o(1), \qquad |\boldsymbol{x}| \to \infty. \tag{2.5}$$

Here $\boldsymbol{G} = \{\boldsymbol{x}\colon\ \boldsymbol{x}' \in \overline{G},\ x_3 = 0\}$. In this case, the convergence is uniform outside any fixed neighborhood of the surface $\boldsymbol{G}$. For the examined periodic set $\boldsymbol{\Gamma}_\varepsilon$, the quantity c_Γ has the form

$$c_\Gamma = \operatorname{cap}(\boldsymbol{\omega})/l^2, \tag{2.6}$$

where $\operatorname{cap}(\boldsymbol{\omega})$ is the harmonic capacity of the set $\boldsymbol{\omega} = \{\boldsymbol{x}\colon\ \boldsymbol{x}' \in \overline{\omega},\ x_3 = 0\}$.

The solution $u^0(\boldsymbol{x})$ of the averaged problem (2.3)–(2.5) satisfies the evenness condition for the variable x_3, i.e., $u^0(\boldsymbol{x}', x_3) = u^0(\boldsymbol{x}', -x_3)$, since relations (2.3)–(2.5) are invariant under replacement of x_3 by $-x_3$. Thus, the function $u^0(\boldsymbol{x})$, which vanishes at infinity and is harmonic in the half-space $\mathbb{R}^3_+$, satisfies the following relations on its boundary:

$$-\frac{\partial u^0}{\partial x_3}(\boldsymbol{x}', 0+) = 2\pi c_\Gamma (1 - u^0(\boldsymbol{x}', 0)), \qquad \boldsymbol{x}' \in G; \tag{2.7}$$

$$\frac{\partial u^0}{\partial x_3}(\boldsymbol{x}', 0+) = 0, \qquad \boldsymbol{x}' \in \mathbb{R}^2 \setminus \overline{G}.$$

The following asymptotic formula is valid

$$u^0(\boldsymbol{x}) = \boldsymbol{c}^0 |\boldsymbol{x}|^{-1} + O(|\boldsymbol{x}|^{-2}), \qquad |\boldsymbol{x}| \to \infty.$$

The constant $\boldsymbol{c}^0$ has the form

$$\boldsymbol{c}^0 = -\frac{1}{2\pi} \iint\limits_G \frac{\partial u^0}{\partial x_3}(\boldsymbol{x}', 0+)\, d\boldsymbol{x}'. \tag{2.8}$$

Using Green's formula

$$\iiint\limits_{\mathbb{R}^3_+} |\nabla_x u^0(\boldsymbol{x})|^2\, d\boldsymbol{x} = -\iint\limits_G u^0(\boldsymbol{x}', 0)\, \frac{\partial u^0}{\partial x_3}(\boldsymbol{x}', 0+)\, d\boldsymbol{x}',$$

from relation (2.8) we obtain the representation

$$\boldsymbol{c}^0 = \frac{1}{2\pi} \iiint\limits_{\mathbb{R}^3_+} |\nabla_x u^0(\boldsymbol{x})|^2\, d\boldsymbol{x} - \frac{1}{2\pi} \iint\limits_G (1 - u^0(\boldsymbol{x}', 0))\, \frac{\partial u^0}{\partial x_3}(\boldsymbol{x}', 0+)\, d\boldsymbol{x}'. \tag{2.9}$$

Since or reasonably large values of $|\boldsymbol{x}|$, the solution $u^\varepsilon(\boldsymbol{x})$ of the starting problem (1.2)–(1.5) converges uniformly to the solution $u^0(\boldsymbol{x})$ of the averaged problem (2.3)–(2.5) as $\varepsilon \to 0$ f, the following limiting relation holds:

$$\lim_{\varepsilon \to 0} \boldsymbol{c}^\varepsilon = \boldsymbol{c}^0.$$

It is worth noting that formula (2.8) can be regarded as the result of the passage to the limit in the first integral in formula (1.10), whereas it is obvious that relation (2.9) cannot be derived by setting $\varepsilon = 0$ in the second integral in formula (1.10). We note that the solution of problem (1.2)–(1.5) satisfies the equality

$$(1 - u^\varepsilon(\boldsymbol{x}', 0))\, \frac{\partial u^\varepsilon}{\partial x_3}(\boldsymbol{x}', 0+) = 0, \qquad \boldsymbol{x}' \in G.$$

The reason is that the function $u^\varepsilon(\boldsymbol{x})$ does not converge to the function $u^0(\boldsymbol{x})$ in the energy norm (see [8, p. 134]).

3. Contact Pressure. The solution of problem (1.2)–(1.5) can be represented in the form of a single-layer potential (see, e.g., [5])

$$u^\varepsilon(\boldsymbol{x}) = \frac{1}{2\pi} \sum_{i,j} \iint\limits_{\omega^{ij}(\varepsilon)} \frac{\varphi^{ij}_\varepsilon(y_1, y_2)\, dy_1\, dy_2}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2}}, \tag{3.1}$$

where the summation is performed over the subscripts i and j for which $\omega^{ij}(\varepsilon) \subset \Gamma_\varepsilon$. In this case, the limiting value of the normal derivative is given by

$$-\frac{\partial u^\varepsilon}{\partial x_3}(\boldsymbol{x}', 0+) = \varphi_\varepsilon^{ij}(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in \omega^{ij}(\varepsilon).$$

Consequently, by virtue of inequality (1.9), the densities $\varphi_\varepsilon^{ij}(\boldsymbol{x}')$ of the integrals in the sum in (3.1) are positive.

The function $\varphi_\varepsilon^{ij}(\boldsymbol{x}')$ coincides [with accuracy up to the factor $[2(1-\nu^2)]^{-1}E$] with the density of the contact pressure $p^\varepsilon(x_1, x_2)$ distributed over the contact spot $\omega^{ij}(\varepsilon)$ [see formula (1.8)]. Accordingly, the quantity

$$F_\varepsilon^{ij} = \iint\limits_{\omega^{ij}(\varepsilon)} \varphi_\varepsilon^{ij}(\boldsymbol{y})\, d\boldsymbol{y} \tag{3.2}$$

defines the force

$$P_\varepsilon^{ij} = \frac{E}{2(1-\nu^2)}\, F_\varepsilon^{ij}, \tag{3.3}$$

exerted on the punch $\omega^{ij}(\varepsilon)$ with accuracy up to the above-mentioned factor.

According to the boundary condition (1.3), the densities $\varphi_\varepsilon^{ij}(\boldsymbol{x}')$ satisfy the system of integral equations of the first kind

$$(B_\varepsilon^{ij}\varphi_\varepsilon^{ij})(\boldsymbol{x}') + {\sum_{k,l}}' (B_\varepsilon^{kl}\varphi_\varepsilon^{kl})(\boldsymbol{x}') = 1, \qquad \boldsymbol{x}' \in \omega^{ij}(\varepsilon). \tag{3.4}$$

Here the summation is performed over the subscripts k and l for which $\omega^{kl}(\varepsilon) \subset \Gamma_\varepsilon$, the prime at the sum indicates that $(k, l) \neq (i, j)$, and B_ε^{ij} is an integral operator defined by

$$(B_\varepsilon^{ij}\varphi)(\boldsymbol{x}') = \frac{1}{2\pi} \iint\limits_{\omega^{ij}(\varepsilon)} \frac{\varphi(\boldsymbol{y})\, d\boldsymbol{y}}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}}. \tag{3.5}$$

To estimate quantity (3.2), we use Mossakovskii's theorem [13], namely: we denote by $\hat{\varphi}_\varepsilon^{ij}(\boldsymbol{x}')$ the solution of the contact problem of a single punch $\omega^{ij}(\varepsilon)$ indented to unit depth, i.e., the solution of the integral equation $(B_\varepsilon^{ij}\hat{\varphi}_\varepsilon^{ij})(\boldsymbol{x}') = 1$ for $\boldsymbol{x}' \in \omega^{ij}(\varepsilon)$. We multiply both sides of Eq. (3.3) by the density $\hat{\varphi}_\varepsilon^{ij}(\boldsymbol{x}')$ and integrate over the site $\omega^{ij}(\varepsilon)$. By virtue of the symmetric kernel of operator (3.5), we obtain

$$\iint\limits_{\omega^{ij}(\varepsilon)} (B_\varepsilon^{ij}\varphi_\varepsilon^{ij})(\boldsymbol{y})\hat{\varphi}_\varepsilon^{ij}(\boldsymbol{y})\, d\boldsymbol{y} = \iint\limits_{\omega^{ij}(\varepsilon)} \varphi_\varepsilon^{ij}(\boldsymbol{y})(B_\varepsilon^{ij}\hat{\varphi}_\varepsilon^{ij})(\boldsymbol{y})\, d\boldsymbol{y}.$$

As a result, we arrive at the equality

$$F_\varepsilon^{ij} + {\sum_{k,l}}' \iint\limits_{\omega^{ij}(\varepsilon)} (B_\varepsilon^{kl}\varphi_\varepsilon^{kl})(\boldsymbol{y})\hat{\varphi}_\varepsilon^{ij}(\boldsymbol{y})\, d\boldsymbol{y} = 2\pi \boldsymbol{c}_\varepsilon^{ij}, \tag{3.6}$$

where $\boldsymbol{c}_\varepsilon^{ij}$ is the translational capacity of the punch $\omega^{ij}(\varepsilon)$, i.e.,

$$\boldsymbol{c}_\varepsilon^{ij} = \frac{1}{2\pi} \iint\limits_{\omega^{ij}(\varepsilon)} \hat{\varphi}_\varepsilon^{ij}(\boldsymbol{y})\, d\boldsymbol{y}. \tag{3.7}$$

Finally, taking into account the positiveness of the densities $\varphi_\varepsilon^{kl}(\boldsymbol{x}')$ for $\boldsymbol{x}' \in \omega^{kl}(\varepsilon)$ and $\hat{\varphi}_\varepsilon^{ij}(\boldsymbol{x}')$ for $\boldsymbol{x}' \in \omega^{ij}(\varepsilon)$ and the positiveness of the kernel of operator (3.5), from relation (3.6) we obtain the estimate

$$F_\varepsilon^{ij} < 2\pi \boldsymbol{c}_\varepsilon^{ij}. \tag{3.8}$$

According to the adopted assumptions on the set Γ_ε, the following limiting relation holds for any segment g of the domain G:

$$\lim_{\varepsilon \to 0} \sum_{(g)} \boldsymbol{c}_\varepsilon^{ij} = |g| c_\Gamma. \tag{3.9}$$

Here $|g|$ is the area of the site g, the sum $\sum_{(g)}$ is taken over those contact spots $\omega^{ij}(\varepsilon)$ that lie inside the domain g and c_Γ is the quantity defined by formula (2.6). Indeed, to prove relation (3.8), one should take into account the equalities

$$\boldsymbol{c}_\varepsilon^{ij} = \varepsilon^2 \operatorname{cap}(\boldsymbol{\omega}), \qquad \sum_{(g)} \boldsymbol{c}_\varepsilon^{ij} = N_\varepsilon \varepsilon^2 \operatorname{cap}(\boldsymbol{\omega}), \tag{3.10}$$

where N_ε is the number of contact spots that lie within the domain g and $N_\varepsilon \sim |g|(\varepsilon^2 l^2)^{-1}$.

Thus, with allowance for inequality (3.7) and condition (3.8), we obtain

$$\lim_{\varepsilon\to 0} \sum_{(g)} F_\varepsilon^{ij} \leqslant 2\pi |g| c_\Gamma. \tag{3.11}$$

Therefore (see [8, § 1]), the densities $\varphi_\varepsilon^{ij}(\boldsymbol{x}')$ and $\boldsymbol{x}' \in \Gamma_\varepsilon$ converge slowly to the limited density $\varphi_0(\boldsymbol{x}')$ distributed over the surface G. In this case, for a fixed point $\boldsymbol{x} \in \mathbb{R}^3_+$, we obtain

$$\lim_{\varepsilon\to 0} u^\varepsilon(\boldsymbol{x}) = \frac{1}{2\pi} \iint_G \frac{\varphi_0(\boldsymbol{y})\, d\boldsymbol{y}}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2}}$$

and the limit

$$\lim_{\varepsilon\to 0} \sum_{(g)} F_\varepsilon^{ij} = \iint_g \varphi_0(\boldsymbol{y})\, d\boldsymbol{y} \tag{3.12}$$

exists for any part g of the region G.

Formula (3.12) shows the mechanical sense of the function $\varphi_0(\boldsymbol{x}')$, namely, the function

$$p^0(x_1, x_2) = \frac{E}{2(1-\nu^2)} \varphi_0(x_1, x_2) \qquad [x_1, x_2) \in G] \tag{3.13}$$

is the averaged contact pressure. In other words, for any segment g of the domain G, we have the limiting relation

$$\iint_g p^0(\boldsymbol{y})\, d\boldsymbol{y} = \lim_{\varepsilon\to 0} \sum_{(g)} \iint_{\omega^{ij}(\varepsilon)} p^\varepsilon(\boldsymbol{y})\, d\boldsymbol{y}. \tag{3.14}$$

However, the pressure from the punch Γ_ε to the surface of the elastic half-space is transmitted through the contact spots $\omega^{ij}(\varepsilon)$. Aa approximate expression for the contact pressure on the contact spot $\omega^{ij}(\varepsilon)$ can be obtained using the solution $\hat{\varphi}_\varepsilon^{ij}(\boldsymbol{x}')$ of the contact problem of a single punch. The reason for this is that as $\varepsilon \to 0$, the relative distance between neighboring punches increases without bound (compared to their diameters).

Thus, using relation (3.12) for the punch $\omega^{ij}(\varepsilon)$ with center at the point (x_1^i, x_2^j) with the coordinates $x_1^i = i\varepsilon l$ and $x_2^j = j\varepsilon l$, we have

$$\frac{F_\varepsilon^{ij}}{\varepsilon^2 l^2} \simeq \varphi_0(x_1^i, x_2^j), \qquad \varepsilon \to 0. \tag{3.15}$$

From formula (3.7), the total load transmitted from the single punch $\omega^{ij}(\varepsilon)$ to the elastic half-space is given by

$$\hat{P}_\varepsilon^{ij} = \frac{E}{2(1-\nu^2)} \hat{F}_\varepsilon^{ij},$$

where

$$\hat{F}_\varepsilon^{ij} = \iint_{\omega^{ij}(\varepsilon)} \hat{\varphi}_\varepsilon^{ij}(\boldsymbol{y})\, d\boldsymbol{y} = 2\pi \boldsymbol{c}_\varepsilon^{ij}. \tag{3.16}$$

Hence, the density $(2\pi\boldsymbol{c}_\varepsilon^{ij})^{-1} \hat{F}_\varepsilon^{ij} \hat{\varphi}_\varepsilon^{ij}(\boldsymbol{x}')$ corresponds to the total load P_ε^{ij} defined by formula (3.3). Therefore, the contact pressure on the contact spot $\omega^{ij}(\varepsilon)$ is given by

$$p^\varepsilon(x_1, x_2) \simeq \frac{E}{2(1-\nu^2)} \frac{F_\varepsilon^{ij}}{2\pi \boldsymbol{c}_\varepsilon^{ij}} \hat{\varphi}_\varepsilon^{ij}(x_1, x_2), \qquad (x_1, x_2) \in \omega^{ij}(\varepsilon). \tag{3.17}$$

Taking into account relations (3.10), (3.13), and (3.15), we finally obtain

$$p^\varepsilon(x_1, x_2) \simeq \frac{l^2}{2\pi \operatorname{cap}(\boldsymbol{\omega})} p^0(x_1^i, x_2^j) \hat{\varphi}_\varepsilon^{ij}(x_1, x_2), \qquad (x_1, x_2) \in \omega^{ij}(\varepsilon), \tag{3.18}$$

where $\hat{\varphi}_\varepsilon^{ij}(\boldsymbol{x}')$ is the solution of the integral equation $(B_\varepsilon^{ij} \hat{\varphi}_\varepsilon^{ij})(\boldsymbol{x}') = 1$ for $\boldsymbol{x}' \in \omega^{ij}(\varepsilon)$.

4. Equation for Determining the Averaged Contact Pressure. To obtain an approximate solution of the integral equations (3.4), we use the localization method [12]. The second term on the left side of Eq. (3.4), which describes the effect of the punches of the system Γ_ε on the contact-pressure distribution under the punch ω_ε^{ij} is approximated by averaging the contact pressure as follows:

$$(B_\varepsilon^{ij}\varphi_\varepsilon^{ij})(\boldsymbol{x}') + \frac{1}{2\pi}\iint\limits_{G\setminus K_\varepsilon^{ij}} \frac{\varphi_0(\boldsymbol{y})\,d\boldsymbol{y}}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}} = 1, \qquad \boldsymbol{x}' \in \omega^{ij}(\varepsilon). \tag{4.1}$$

Here K_ε^{ij} is a square with side εl and center at the point (x_1^i, x_2^j).

Next, according to formula (3.17),

$$\varphi_\varepsilon^{ij}(\boldsymbol{x}') = \frac{F_\varepsilon^{ij}}{2\pi \boldsymbol{c}_\varepsilon^{ij}}\,\hat\varphi_\varepsilon^{ij}(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in \omega^{ij}(\varepsilon).$$

Substituting this expression into Eq. (4.1), with accuracy up to terms of order ε, we obtain

$$\frac{F_\varepsilon^{ij}}{2\pi \boldsymbol{c}_\varepsilon^{ij}} + \frac{1}{2\pi}\iint\limits_{G} \frac{\varphi_0(\boldsymbol{y})\,d\boldsymbol{y}}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}} = 1, \qquad \boldsymbol{x}' \in \omega^{ij}(\varepsilon).$$

Taking into account relation (3.15), we have

$$\frac{\varepsilon^2 l^2}{2\pi \boldsymbol{c}_\varepsilon^{ij}}\,\varphi_0(x_1^i, x_2^j) + \frac{1}{2\pi}\iint\limits_{G} \frac{\varphi_0(\boldsymbol{y})\,d\boldsymbol{y}}{\sqrt{(x_1^i-y_1)^2+(x_2^j-y_2)^2}} = 1. \tag{4.2}$$

Because the point (x_1^i, x_2^j) is taken arbitrarily, using the first formula (3.10) and Eq. (4.2), we obtain

$$\frac{l^2}{2\pi\,\mathrm{cap}\,(\boldsymbol{\omega})}\,\varphi_0(x_1,x_2) + \frac{1}{2\pi}\iint\limits_{G} \frac{\varphi_0(\boldsymbol{y})\,d\boldsymbol{y}}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}} = 1. \tag{4.3}$$

Thus, according to formulas (4.3) and (2.6), the averaged contact-pressure density (3.13) satisfies the integral equation

$$\frac{1}{c_\Gamma}\,p^0(x_1,x_2) + \iint\limits_{G} \frac{p^0(\boldsymbol{y})\,d\boldsymbol{y}}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}} = \frac{\pi E}{1-\nu^2}. \tag{4.4}$$

It is easy to see that Eq. (4.4) or, what is the same, Eq. (4.3) is equivalent to boundary condition (2.7).

The positiveness condition for the contact pressure on the contact spots (1.9) and the limiting relation (3.14) with allowance for (3.9) lead to the inequality

$$p^0(x_1,x_2) > 0, \qquad (x_1,x_2)\in G. \tag{4.5}$$

Therefore, according to relations (4.3) and (4.5), the quantity $u^0(x_1,x_2,0)$, which is interpreted as the averaged displacement of the elastic-foundation surface under the punch with the fine-grained base Γ_ε is smaller than the punch displacement, i.e.,

$$u^0(x_1,x_2,0) < 1, \qquad (x_1,x_2)\in G.$$

Hence, the second term in formula (2.9) is positive.

5. Generalizations and Remarks. 1. For a punch Γ_ε with a nonflat base, boundary condition (1.3) should be replaced by the relation

$$u^\varepsilon(\boldsymbol{x}',0) = w(\boldsymbol{x}'), \qquad \boldsymbol{x}' = (x_1,x_2)\in\Gamma_\varepsilon. \tag{5.1}$$

Here $w(x_1,x_2)$ is a smooth function that defines the displacement of the elastic-foundation surface under the punch. In particular, the function $w(\boldsymbol{x}') = \delta_0 - \beta_2 x_1 + \beta_1 x_2$ corresponds to an inclined punch with a fine-grained flat base (δ_0 is the translational displacement and β_1 and β_2 are the angles of rotation about horizontal axes).

According to the Marchenko–Khruslov theory (see [8, Theorem 1.4]), the boundary condition (5.1) corresponds to the following averaged boundary condition [instead of (2.7)]:

$$-\frac{\partial u^0}{\partial x_3}(\boldsymbol{x}',0+) = 2\pi c_\Gamma(w(\boldsymbol{x}') - u^0(\boldsymbol{x}',0)), \qquad \boldsymbol{x}'\in G. \tag{5.2}$$

In this case, the averaged contact-pressure density is found by solving the integral equation [instead of Eq. (4.4)]

$$\frac{1}{c_\Gamma}\, p^0(x_1,x_2)+\iint\limits_G \frac{p^0(\boldsymbol{y})\,d\boldsymbol{y}}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}}=\frac{\pi E}{1-\nu^2}\, w(x_1,x_2). \tag{5.3}$$

For the integral equation (5.3), various methods of solution have been proposed (see a survey [14, § 3.5.1]). It is clear that for an arbitrary right side of Eq. (5.3), its solution does not satisfy the positiveness condition (4.5). In addition, the solution of the integral equation (4.4) has no singularity at the boundary of the domain G.

2. The averaged contact problem (5.3) remains valid if the contact spots $\omega^{ij}(\varepsilon)$ in the set Γ_ε are expanded arbitrarily. Moreover, the results of the theory of [7, 8] are valid for weaker constraints on the structure of the fine-grained boundary Γ_ε. For example, we can discard the assumption of periodicity and take into account the dependence of the contact spots on slow variables. The assumption of periodicity was used to simplify the calculation of the coefficient c_Γ.

In the particular case where the contact spots $\omega^{ij}(\varepsilon)$ are located at the nodes of an oblique grid with sides εl_1 and εl_2 and an angle α, we obtain

$$c_\Gamma=\operatorname{cap}(\boldsymbol{\omega})/(l_1 l_2\sin\alpha). \tag{5.4}$$

For a hexagonal grid with an internodal distance εl, we have

$$c_\Gamma=\frac{8}{3\sqrt{3}}\,\frac{\operatorname{cap}(\boldsymbol{\omega})}{l^2}. \tag{5.5}$$

For the contact spots of the two types located periodically at the nodes of a square grid with side εl in staggered order, we obtain

$$c_\Gamma=(\operatorname{cap}(\boldsymbol{\omega}_1)+\operatorname{cap}(\boldsymbol{\omega}_2))/(2l^2). \tag{5.6}$$

Formulas (5.4)–(5.6) follow from relation (3.9).

3. The resulting equation (5.3) was derived using potential theory. To extend the formulation of the contact problem of a punch with a fine-grained base, for example, to the case of an elastic layer, we rewrite the averaged problem in terms of linear elastic theory.

Thus, the formulation of the contact problem of a frictionless punch Γ_ε pressed into an elastic half-space includes the boundary conditions

$$\sigma_{31}(\boldsymbol{U}^\varepsilon;\boldsymbol{x}',0)=\sigma_{32}(\boldsymbol{U}^\varepsilon;\boldsymbol{x}',0)=0,\qquad \boldsymbol{x}'\in\mathbb{R}^2;$$

$$U_3^\varepsilon(\boldsymbol{x}',0)=w(\boldsymbol{x}'),\qquad \boldsymbol{x}'\in\Gamma_\varepsilon;$$

$$\sigma_{33}(\boldsymbol{U}^\varepsilon;\boldsymbol{x}',0)=0,\qquad \boldsymbol{x}'\in\mathbb{R}^2\setminus\overline{\Gamma}_\varepsilon.$$

Here $\sigma_{3i}(\boldsymbol{U}^\varepsilon)$ are the stress-tensor components. In this case, the contact pressure under the punch is given by

$$p^\varepsilon(\boldsymbol{x}')=-\sigma_{33}(\boldsymbol{U}^\varepsilon;\boldsymbol{x}',0),\qquad \boldsymbol{x}'\in\Gamma_\varepsilon.$$

The solution $\boldsymbol{U}^0(\boldsymbol{x})$ of the averaged problem should satisfy the system of Lamé differential equations in a semi-infinite domain occupied by an elastic body, the frictionless boundary conditions

$$\sigma_{31}(\boldsymbol{U}^0;\boldsymbol{x}',0)=\sigma_{32}(\boldsymbol{U}^0;\boldsymbol{x}',0)=0,\qquad \boldsymbol{x}'\in\mathbb{R}^2,$$

the condition of zero load increment outside the domain occupied by the punch, i.e.,

$$\sigma_{33}(\boldsymbol{U}^0;\boldsymbol{x}',0)=0,\qquad \boldsymbol{x}'\in\mathbb{R}^2\setminus\overline{G},$$

and the averaged contact condition

$$-\sigma_{33}(\boldsymbol{U}^0;\boldsymbol{x}',0)=\frac{\pi E}{1-\nu^2}\,c_\Gamma(w(\boldsymbol{x}')-U_3^0(\boldsymbol{x}',0)),\qquad \boldsymbol{x}'\in G. \tag{5.7}$$

We note that boundary condition (5.7) is condition (5.2) rewritten with allowance for the relations

$$\sigma_{33}(\boldsymbol{U}^0;\boldsymbol{x}',0)=\frac{E}{2(1-\nu^2)}\,\frac{\partial u^0}{\partial x_3}(\boldsymbol{x}',0),\qquad U_3^0(\boldsymbol{x}',0)=u^0(\boldsymbol{x}',0),$$

which follow from Belyaev's formulas (1.6) and (1.7).

Thus, the averaged contact boundary condition (5.7) is free from the assumption that the elastic body occupies half-space and can be used to solve the contact problem of a punch with a fine-grained base pressed into the flat boundary of an elastic foundation in the case of an elastic layer, plate, etc.

4. The assumption that the complement to the set Γ_ε is connected in the case (1.1) is also of no significance in the present approach. For example, an approximate solution of the contact problem of a so-called netlike punch can be obtained in a similar manner (see example 2 in [8, Ch. 1, § 4]).

5. Using formula (2.8), we obtain

$$\boldsymbol{c}^0 = -\frac{1}{2\pi} \iint\limits_G \varphi_0(\boldsymbol{y})\, d\boldsymbol{y},$$

where $\varphi(\boldsymbol{x}')$ is a solution of Eq. (4.3).

With allowance for relation (3.12), we have

$$\iint\limits_G \varphi_0(\boldsymbol{y})\, d\boldsymbol{y} = \lim_{\varepsilon\to 0} \sum_{(G)} F_\varepsilon^{ij},$$

which agrees with the first formula (1.10) using notation (3.3).

Integration of both sides of Eq. (4.3) over the domain G yields

$$\frac{1}{2\pi} \iint\limits_G \varphi_0(\boldsymbol{y})\, d\boldsymbol{y} + \frac{c_\Gamma}{2\pi} \iint\limits_G \varphi_0(\boldsymbol{y}) \iint\limits_G \frac{d\boldsymbol{x}'\, d\boldsymbol{y}}{|\boldsymbol{x}' - \boldsymbol{y}|} = |G| c_\Gamma. \tag{5.8}$$

Using formula (3.9), we write the right side of Eq. (5.8) as

$$|G| c_\Gamma = \lim_{\varepsilon\to 0} \sum_{(G)} \boldsymbol{c}_\varepsilon^{ij}. \tag{5.9}$$

Comparing formula (5.4) with (5.8) and (5.9), we infer that the limiting value $\boldsymbol{c}^0$ of the translational capacity $\boldsymbol{c}^\varepsilon$ is smaller than the total capacity of the punches in the system. Obviously, this is also true for the capacity $\boldsymbol{c}^\varepsilon$.

We set

$$m_G = \min_{\boldsymbol{y}\in G} \iint\limits_G \frac{d\boldsymbol{x}'}{|\boldsymbol{x}' - \boldsymbol{y}|}, \qquad M_G = \max_{\boldsymbol{y}\in G} \iint\limits_G \frac{d\boldsymbol{x}'}{|\boldsymbol{x}' - \boldsymbol{y}|}.$$

From formula (5.8) follow the estimates

$$\frac{|G| c_\Gamma}{1 + M_G c_\Gamma} \leqslant \boldsymbol{c}^0 \leqslant \frac{|G| c_\Gamma}{1 + m_G c_\Gamma}.$$

We note that for a circular region G, the quantity M_G exceeds m_G by a factor of $\pi/2$ (see, e.g., [14, § 1.1.6]).

6. The resulting integral equation (4.4) of the averaged contact problem was obtained by the localization method. Therefore, for the contact problem considered, the localization method can be rigorously substantiated within the averaging theory [8]. We also note that estimate (3.10) was obtained by a method different from that of [8].

7. Let us compare the results obtained above with those of [9]. First, the equations for determining the averaged contact-pressure density differ radically, namely: in the case (1.1), Eq. (4.4) differs from the corresponding equation for the case (1.12) by the presence of the first term.

Second, one can easily see that the asymptotic representation (3.18) for the true contact pressure is similar to that obtained earlier (see [9, formula (25)]). In both cases, the representation has the form

$$p^\varepsilon(x_1, x_2) \simeq p^0(x_1, x_2) f_\varepsilon^{ij}(x_1, x_2), \qquad (x_1, x_2) \in \omega^{ij}(\varepsilon). \tag{5.10}$$

Here $p^0(x_1, x_2)$ is the averaged contact pressure at the center of the contact spot $\omega^{ij}(\varepsilon)$ and $f_\varepsilon^{ij}(x_1, x_2)$ is a function that describes the pressure distribution over the contact spot $\omega^{ij}(\varepsilon)$. We also note that in both cases, the function $f_\varepsilon^{ij}(x_1, x_2)$ has a root singularity on the boundary of the contact spot $\omega^{ij}(\varepsilon)$.

In the case considered, from formulas (3.18) and (3.16) we obtain

$$f_\varepsilon^{ij}(x_1,x_2) = \frac{l^2}{2\pi\,\mathrm{cap}\,(\boldsymbol{\omega})}\,\hat{\varphi}_\varepsilon^{ij}(x_1,x_2),$$

$$\frac{1}{\varepsilon^2 l^2}\iint\limits_{\omega^{ij}(\varepsilon)} f_\varepsilon^{ij}(x_1,x_2)\,dx_1dx_2 = 1. \tag{5.11}$$

The quantity $\varepsilon^2 l^2$ is equal to the area of the periodicity cell

$$K_\varepsilon^{ij} = \{\boldsymbol{x}' :\ \varepsilon^{-1}(x_1 - i\varepsilon l, x_2 - j\varepsilon l) \in K\}.$$

In the case of (1.12), the normalization condition (5.11) follows from formulas (23) and (28) of [9].

Taking into account the relation

$$\iint\limits_{g\cap\Gamma_\varepsilon} p^\varepsilon(\boldsymbol{y})\,d\boldsymbol{y} \simeq \iint\limits_{g} p^0(\boldsymbol{y})\,d\boldsymbol{y},$$

which is valid for any segment g of the domain G and normalization condition (5.11), from formula (5.10) we obtain

$$\iint\limits_{g} p^0(\boldsymbol{y})\,d\boldsymbol{y} \simeq \iint\limits_{g} p^0(\boldsymbol{y}) f_\varepsilon^{ij}(\boldsymbol{y})\,d\boldsymbol{y}. \tag{5.12}$$

In (5.12), it is understood that $f_\varepsilon^{ij}(\boldsymbol{y}) = 0$ for $\boldsymbol{y} \in K_\varepsilon^{ij} \setminus \overline{\omega^{ij}(\varepsilon)}$.

We note that in the mechanics of rough elastic bodies in contact (see, e.g., [15]), the function $f_\varepsilon^{ij}(x_1,x_2)$ is called a local contact intensity factor.

8. Finally, we note that in discrete contact problems, the number of punches N is usually a primary parameter and the derivative parameter ε is calculated based on the geometry of punch location. In the case (1.1), the inversion of (1.13) yields

$$\varepsilon \sim \sqrt{|G|/(Nl^2)},$$

where l is the characteristic dimension of the domain G. In this case, the coefficient c_Γ appearing in the averaged problem and determined by formula (2.6) should be calculated by the formula

$$c_\Gamma = \boldsymbol{c}^\varepsilon/(\varepsilon^2 l^2) = N\boldsymbol{c}^\varepsilon/|G|.$$

Here $\boldsymbol{c}_\varepsilon = \boldsymbol{c}_\varepsilon^{ij}$ is the translational capacity of the punch $\omega^{ij}(\varepsilon)$ defined by formula (3.7).

We are grateful to T. A. Mel'nik for the formulation of the problem and discussions of the results.

This work was supported by the Ministry of Industry, Science, and Technology of the Russian Federation (Grant No. MD-182.2003.01).

REFERENCES

1. L. A. Galin, *Contact Problems of the Theory of Elasticity and Viscoelasticity* [in Russian], Nauka, Moscow (1980).
2. N. M. Belyaev, "Local stresses in compressed elastic bodies," in: *Engineering Structures and Structural Mechanics* [in Russian], Put', Leningrad (1924), pp. 27–74.
3. L. Bers, F. John, and M. Schechter, *Partial Differential Equations*, Interscience, New York (1964).
4. G. Pólya and G. Szegö, *Isoparametric Inequalities in Mathematical Physics*, Princeton Univ. Press, Princeton (1951).
5. N. S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, Berlin (1972).
6. I. I. Argatov, "Capacity characteristics of a punch with a smooth flat end," *Izv. Vyssh. Uch. Zaved. Stroit. Arkhitektura*, No. 4, 26–32 (2000).
7. V. A. Marchenko and E. Ya. Khruslov "Boundary-value problems with fine-grain boundaries," *Mat. Sb.*, **65**, No. 3, 458–472 (1964),

8. V. A. Marchenko and E. Ya. Khruslov, *Boundary-Value Problems in Domains with Fine-Grain Boundaries* [in Russian], Naukova Dumka, Kiev (1974).
9. I. I. Argatov and T. A. Mel'nyk, "Homogenization of a contact problem for a system of densely situated punches," *Europ. J. Mech. A. Solids*, **20**, No. 1, 91–98 (2001).
10. G. M. L. Gladwell and V. I. Fabrikant, "The interaction between a system of circular punches on an elastic half-space," *Trans. ASME, J. Appl. Mech.*, **49**, No. 2, 341–344 (1982).
11. I. I. Argatov, "Interaction between punches on an elastic half-space," *Usp. Mekh.*, **1**, No. 4, 8–40 (2002).
12. I. G. Goryacheva, *Mechanics of Friction Interaction* [in Russian], Nauka, Moscow (2001).
13. V. I. Mossakovskii, "Estimating displacements in spatial contact problems," *Prikl. Mat. Mekh.*, **15**, No. 5, 635–636 (1951).
14. I. I. Argatov and N. N. Dmitriev, *Fundamentals of Theory of Elastic Discrete Contact* [in Russian], Politekhnika, St. Petersburg (2003).
15. F. D. Fischer, W. Daves, and E. A. Werner, "On the temperature in the wheel-rail rolling contact," *Fatigue Fract. Eng. Mater. Struct.* **26**, 999–1006 (2003).